

**Modified Enlarged 18pt**

**OXFORD CAMBRIDGE AND RSA EXAMINATIONS**

**Monday 18 October 2021 – Afternoon**

**A Level Mathematics B (MEI)**

**H640/03 Pure Mathematics and Comprehension**

**Insert**

**Time allowed: 2 hours**

**plus your additional time allowance**



## **INSTRUCTIONS**

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## **INFORMATION**

**This Insert contains the article for Section B.**

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## ADDING ARCTANGENTS

Where does the name ‘arctangent’ come from?

The two commonly used ways to denote the angle which has a tangent  $x$  are  $\tan^{-1}x$  and  $\arctan x$ . The first of these is related to inverse function notation,  $f^{-1}(x)$ . Arctangent comes from radian measure, where an angle is represented by an arc on a unit circle;  $\arctan x$  is the arc whose tangent is  $x$ .

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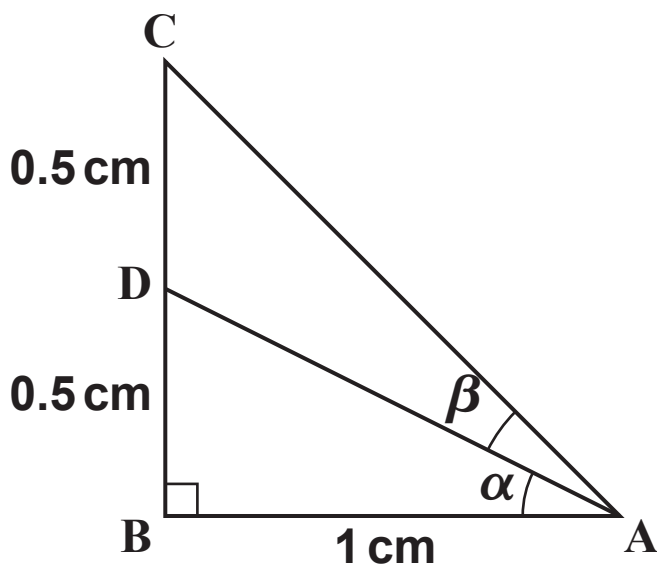
An interesting result

It can be shown that  $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan 1$ .

Consider the diagram in FIG. C1.

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FIG. C1



Triangle ABC is right-angled at B.

$AB = BC = 1 \text{ cm}$ .

D is the midpoint of BC.

Using triangle ABD,  $\tan\alpha = \frac{DB}{BA} = \frac{1}{2}$  so  $\alpha = \arctan\left(\frac{1}{2}\right)$ .

Using triangle ABC,  $\tan(\alpha + \beta) = 1$  so  $\alpha + \beta = \arctan 1$ . 15

Hence  $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = 1$ .

Using  $\tan\alpha = \frac{1}{2}$  and finding  $\tan\beta$ , it follows that

$$\beta = \arctan\left(\frac{1}{3}\right),$$

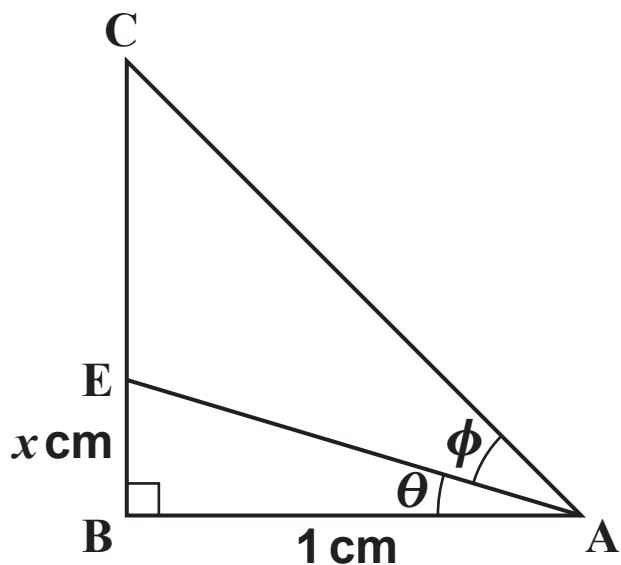
which gives the required result that

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan 1.$$

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## Generalising the result

**FIG. C2**



Triangle ABC in FIG. C2 is the same as triangle ABC in FIG. C1 but E is a point on BC such that  $EB = x$  cm and  $\theta = \arctan x$ .

Following the same method as above,

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$$\arctan x + \arctan\left(\frac{1-x}{1+x}\right) = \arctan 1.$$

## The arctan addition formula

The arctangent addition formula is a further generalization:

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right), \text{ as long as } xy < 1. \quad 30$$

This result is equivalent to the addition formula

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \text{ where } \alpha = \arctan x \text{ and } \beta = \arctan y.$$

To see why the restriction  $xy < 1$  is necessary, consider what happens if  $xy \geq 1$ . 35

Clearly,  $\frac{x+y}{1-xy}$  is undefined when  $xy = 1$ , so the formula does not apply in this case.

Suppose next that  $xy > 1$ , and that  $x$  and  $y$  are both positive; in this case  $y > \frac{1}{x}$ .

For any positive  $x$ ,  $\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$ . 40

$y > \frac{1}{x} \Rightarrow \arctan y > \arctan\left(\frac{1}{x}\right)$  so it follows that  $\arctan x + \arctan y > \frac{\pi}{2}$ .

However,  $\arctan\left(\frac{x+y}{1-xy}\right)$  cannot be greater than  $\frac{\pi}{2}$  as the range of the arctan function is  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ .

The formula  $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$  therefore cannot be valid in this case. 45

A similar argument can be used to show that the formula cannot be valid when  $xy > 1$  and  $x$  and  $y$  are both negative.

If  $xy > 1$ , the arctangent addition formula needs to be adapted, as shown below. 50

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) - \pi, \text{ when } xy > 1 \text{ and } x, y < 0$$

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) + \pi, \text{ when } xy > 1 \text{ and } x, y > 0$$
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### Some additional results

- For  $n$  a positive integer,  

$$\arctan\left(\frac{1}{n+1}\right) + \arctan\left(\frac{1}{n^2+n+1}\right) = \arctan\left(\frac{1}{n}\right);$$
this follows directly from the arctan addition formula in line 30. 60

- $\arctan 1 + \arctan 2 + \arctan 3 = \pi$ . This can be proved by using  $\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$  together with  $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan 1$ .



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